ADDENDUM TO "SUPERCONNECTIONS AND PARALLEL TRANSPORT"

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Abstract. In this addendum to our article "Superconnections and Parallel Transport" we give an alternate construction to the parallel transport of a superconnection contained in Corollary 4.4 of [2], which has the advantage that is independent on the various ways a superconnection splits as a connection plus a bundle endomorphism valued form.

Consider as in Section 4 of [2] a superconnection $\mathbb{A}$ in the sense of Quillen (see [3] and [1]) on a $\mathbf{Z}/2$-graded vector bundle E over a *manifold* M, i.e. an odd first-order differential operator

$$\mathbb{A} : \Omega^*(M, E) \to \Omega^*(M, E)$$

satisfying Leibniz rule

$$\mathbb{A}(\omega \otimes s) = d\omega \otimes s \pm \omega \otimes \mathbb{A}(s),$$

with $\omega \in \Omega^*(M)$ differential form on M and $s \in \Gamma(M; E)$ arbitrary section of the bundle E over M. For such a superconnection we defined in [2] a notion of parallel transport along (families of) superpaths $c : S \times \mathbf{R}^{1|1} \to M$ that is compatible under glueing of superpaths. Let us briefly recall this construction. First, let us write $\mathbb{A} = \mathbb{A}_1 + A$, with $\mathbb{A}_1 = \nabla$ the connection part of the superconnection $\mathbb{A}$ and $A \in \Omega^*(M, End\ E)^{odd}$ the linear part of the superconnection. For an arbitrary superpath c in M consider the diagram

$$\begin{tikzcd}
E \arrow[dd] & & c^*E \arrow[ll] \arrow[dl] \arrow[dd] \\
& \pi^*E \arrow[ul] \arrow[dd] & \\
M & & S \times \mathbf{R}^{1|1} \arrow[ll, "c"'] \arrow[dl, dashed, "\tilde{c}"] \\
& \Pi TM \arrow[ul, "\pi"] &
\end{tikzcd}$$

with $\tilde{c}$ a canonical lift of the path c to ΠTM, the "odd tangent bundle" of M. Then parallel transport along c is defined by *parallel* sections $\psi \in \Gamma(c^*E)$

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along c which are solutions to the following differential equation

$$(c^*\nabla)_D\psi - (\tilde{c}^*A)\psi = 0.$$

Here $D = \partial_\theta + \theta\partial_t$ denotes the standard (right invariant) vector field on $\mathbf{R}^{1|1}$, see Section 2.4 of [2].

Our alternate construction goes as follows. We first write $\mathbb{A} = \mathbb{A}_0 + \bar{\mathbb{A}}$, where $\mathbb{A}_0$ denotes the zero part of the superconnection and $\bar{\mathbb{A}}$ the remaining part. Define then a connection $\bar{\nabla}$ on the bundle π^*E over ΠTM as follows

$$\bar{\nabla}_{\mathcal{L}_X}(\omega \otimes s) = \mathcal{L}_X\omega \otimes s \pm \iota_X\bar{\mathbb{A}}s,$$
$$\bar{\nabla}_{\iota_X}(\omega \otimes s) = \iota_X\omega \otimes s,$$

for $\omega \in \Omega^*(M)$ and $s \in \Gamma(M;E)$. Here, for a vector field X on M, $\mathcal{L}_X$ and ι_X denote the Lie derivative respectively contraction in the X-direction acting as even respectively odd derivations on $\Omega^*(M) = \mathcal{C}^\infty(\Pi TM)$, i.e. as vector fields on ΠTM. These relations are enough to define a connection $\bar{\nabla}$ on the bundle π^*E over ΠTM since the algebra of vector fields on ΠTM is generated over $\mathcal{C}^\infty(\Pi TM)$ by vector fields of the type $\mathcal{L}_X$ and ι_X, for X arbitrary vector field on M, i.e.

$$Vect(\Pi TM) = \mathcal{C}^\infty(\Pi TM) < \mathcal{L}_X, \iota_X \mid X \in Vect(M) > .$$

Parallel transport along a superpath $c : S \times \mathbf{R}^{1|1} \to M$ is defined by *parallel sections* $\psi \in \Gamma(c^*E)$ along c which are solutions to the following differential equation

$$(\tilde{c}^*\bar{\nabla})_D\psi - (c^*\mathbb{A}_0)\psi = 0.$$

As before, the parallel transport is well-defined (cf. Proposition 4.2 of [2]) by this "half-order" differential equation and is compatible under glueing of superpaths (i.e. it satisfies properties (i) and (ii) of Theorem 4.3 in [2]). The advantage of this construction resides in the fact that the parallel transport so defined is invariant under the various ways in which a superconnection can be written as a sum of a connection plus a linear part, as $\bar{\mathbb{A}}$ is invariant under such splittings.

Denote by D the de Rham differential on ΠTM. If ω is a function on ΠTM, then the 1-form $D\omega$ on ΠTM evaluated on the standard odd vector field d on ΠTM gives us

$$(D\omega)(d) = d\omega,$$

the differential of ω, understood as a function on ΠTM. This allows us to conclude that for any s a section of E,

$$\bar{\nabla}_d s = \bar{\mathbb{A}}s.$$

Let us note that the connection $\bar{\nabla}$ is torsion free in the odd directions, i.e.

$$[\bar{\nabla}_{\iota_X}, \bar{\nabla}_{\iota_Y}] = \bar{\nabla}_{[\iota_X,\iota_Y]}\ (= 0),$$

where X and Y are vector fields on M.

The two constructions coincide when we consider connections instead of superconnections on the bundle E over M. When the manifold M reduces to a point, a graded vector bundle with superconnection reduces to a $\mathbf{Z}/2$-vector space V together with an odd endomorphism A ($= \mathbb{A}_0$) of V. The two constructions of parallel transport we considered also coincide in this situation giving rise to the supergroup homomorphism of Example 3.2.9 in [4]

$$\mathbf{R}^{1|1} \ni (t, \theta) \longmapsto e^{-tA^2+\theta A} \in GL(V),$$

as solution to the "half-order" differential equation $D\psi = A\psi$.

We can also *recover* the superconnection from its associated parallel transport by first recovering the $\mathbb{A}_0$ from looking at constant superpaths in M, and then recover $\bar{\mathbb{A}}$ by looking at parallel transport along the superpath given by $\mathbf{R}^{1|1} \times \Pi TM \to \mathbf{R}^{0|1} \times \Pi TM \to M$, where the two maps are the obvious projection maps. The lift of such a superpath to ΠTM gives, after the obvious projection map, the flow of the vector field d on ΠTM. Given that $\bar{\nabla}_d s = \bar{\mathbb{A}} s$, this recovers $\bar{\mathbb{A}}$. Compare with Section 4.4 of [2].

Acknowledgements. The construction presented here is a mere continuation of an idea of Stephan Stolz who first thought to interpret a Quillen superconnection on a bundle E over M as a connection on the pullback bundle π^*E over ΠTM. I would like to thank Peter Teichner for suggesting to write up this Addendum.

References

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